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BTZ Black Hole in Non-Commutative Spaces

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In this letter we compute the corrections to the horizons, the horizon area and Hawking temperature of a BTZ black hole. These corrections stem from the space non-commutativity. We show that in non-commutative case, non-rotating BTZ black hole in contrast with commutative case has two horizons.

KEY WORDS: non-commutative space; BTZ black hole; hawking temperature.

1. INTRODUCTION

In 1992 Bañados, Teitelboim and Zanelli (BTZ) (Bañados *et al.*, 1992, 1993) showed that $(2 + 1)$-dimensional gravity has a black hole solution. This black hole is described by two parameters, its mass M and its angular momentum (spin) J. It is locally anti-de-Sitter space and thus it differs from Schwarzschild and Kerr solutions in that it is asymptotically anti-de-Sitter instead of flat spacetime. Additionally, it has no curvature singularity at the origin. AdS black holes, which are members of the two-parametric family of BTZ black holes, play a central role in AdS/CFT conjecture (Maldacena, 1998) and also in brane-world scenarios (Randall and Sundrum, 1999a, 1999b). Specifically AdS(2) black hole is most interesting in the context of string theory and black hole physics (Birmingham *et al.*, 1997; Maldacena *et al.*, 1999; Spradlin and Strominger, 1999).

Black hole thermodynamic quantities depend on the Hawking temperature via the usual thermodynamic relations. The Hawking temperature undergoes corrections from many sources:the quantum corrections (Das *et al.*, 2002; Lidsey *et al.*, 2002; Nojiri *et al.*, 2003; Setare, 2003, 2004a), the self-gravitational corrections (Keski-Vakkuri and Kraus, 1996; Parikh and Wilczek, 2000; Setare and Vagenas, 2004), and the corrections due to the generalized uncertainty principle (Setare, 2004b).

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In this letter we concentrate on the corrections due to the space non commutativity. Recently there has been considerable interest in the possible effects of the non commutative space (Seiberg and Witten, 1999). In Ahluwalia (1994) the author have argued that every consideration on space time measurement that allows gravitational effects asks for non-commutative space time.

2. BTZ BLACK HOLES

The black hole solutions of Bañados *et al.* (1992, 1993) in $(2+1)$ spacetime dimensions are derived from a three dimensional theory of gravity

$$S = \int dx^3 \sqrt{-g}\left({}^{(3)}R + 2\Lambda\right) \tag{1}$$

with a negative cosmological constant ($\Lambda = \frac{1}{l^2} > 0$).
The corresponding line element is

$$ds^2 = -\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)dt^2 + \frac{dr^2}{\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)} + r^2\left(d\theta - \frac{J}{2r^2}dt\right)^2 \tag{2}$$

with M the Arnowitt-Deser-Misner (ADM) mass, J the angular momentum (spin) of the BTZ black hole and $-\infty < t < +\infty$, $0 \le r < +\infty$, $0 \le \theta < 2\pi$. The metric is singular when $r = r_\pm$, where

$$r_\pm^2 = \frac{Ml^2}{2}\left[1 \pm \left(1 - \left(\frac{J}{Ml}\right)^2\right)^{1/2}\right] \tag{3}$$

and

$$M = \frac{r_+^2 + r_-^2}{l^2}, \qquad J = \frac{2r_+r_-}{l} \tag{4}$$

Note that $r_\pm$ become complex if $|J| > Ml$ and the horizons disappear. In order to have horizon we must have the following conditions

$$M > 0, \qquad |J| \le Ml \tag{5}$$

The non rotated BTZ black hole can be obtain by putting $J = 0$ in Eq. (2) as following

$$ds^2 = -\left(-M + \frac{r^2}{l^2}\right)dt^2 + \frac{dr^2}{\left(-M + \frac{r^2}{l^2}\right)} + r^2 d\theta^2, \tag{6}$$

which has an horizon at

$$r_+ = \sqrt{\frac{M}{\Lambda}}, \tag{7}$$

and is similar to Schwarzschild black hole with the important difference that it is not asymptotically flat but it has constant negative curvature.

In the case of $M = -1$, $J = 0$ metric may be recognized as that of ordinary ant-de Sitter space, it is seperated by mass gap from $M = 0$, $J = 0$, massless black hole.

The area A and Hawking temperature T_H of the event (outer) horizon are (Carlip, 1995; Carlip and Teitelboim, 1995)

$$A = 2\pi l \left(\frac{M + \sqrt{M^2 + \frac{J^2}{l^2}}}{2} \right)^{1/2} = 2\pi r_+ \tag{8}$$

$$T_H = \frac{1}{2\pi l^2} \left(\frac{r_+^2 - r_-^2}{r_+} \right). \tag{9}$$

3. NON-COMMUTATIVITY AND BTZ BLACK HOLES

We have the following metric for BTZ black hole in non-commutative space

$$ds^2 = -f(\tilde{r})dt^2 + \frac{d\tilde{r}d\tilde{r}}{f(\tilde{r})} + \tilde{r}\tilde{r}\left(d\theta - \frac{J}{2\tilde{r}\tilde{r}}dt\right)^2 \tag{10}$$

with

$$f(\tilde{r}) = \left(-M + \frac{\tilde{r}\tilde{r}}{l^2} + \frac{J^2}{4\tilde{r}\tilde{r}}\right) \tag{11}$$

where $\tilde{r}$ satisfies following Poisson brackets (Li, in press; Nasseri, in press)

$$\{\tilde{x_i}, \tilde{x_j}\} = \theta_{ij}, \qquad \{\tilde{x_i}, \tilde{p_j}\} = \delta_{ij}, \qquad \{\tilde{p_i}, \tilde{p_j}\} = 0. \tag{12}$$

Since the non-commutativity parameter should very small compared to the length scales of the black hole, one can treat the noncommutative effects as some perturbations of the commutative counter-part, $f(\tilde{r})$ in terms of the noncommutative coordinates $\tilde{x_i}$ is as

$$f(\tilde{r}) = \left(-M + \frac{\tilde{x_i}\tilde{x_i}}{l^2} + \frac{J^2}{4\tilde{x_i}\tilde{x_i}}\right) \tag{13}$$

We note that there is a new coordinate system (Chaichian *et al.*, 2001)

$$x_i = \tilde{x_i} + \frac{1}{2}\theta_{ij}\tilde{p_j}, \qquad p_j = \tilde{p_j}, \tag{14}$$

where the new variables satisfy the usual Poisson brackets

$$\{x_i, x_j\} = 0, \qquad \{x_i, p_j\} = \delta_{ij}, \qquad \{p_i, p_j\} = 0. \tag{15}$$

Using the new coordinates, we have

$$f(r) = -M + \frac{\left(x_i - \frac{\theta_{ij}p_j}{2}\right)\left(x_i - \frac{\theta_{ik}p_k}{2}\right)}{l^2} + \frac{J^2}{4\left(x_i - \frac{\theta_{ij}p_j}{2}\right)\left(x_i - \frac{\theta_{ik}p_k}{2}\right)} \tag{16}$$

where $\theta_{ij} = 1/2\varepsilon_{ijk}\theta_k$. The equation

$$f(r_{h\pm}) = 0, \tag{17}$$

where $r_{h\pm}$ is the modified horizon, leads us to

$$f(r_{h\pm}) = -M + \frac{r_{h\pm}^2 - \frac{\vec{L}.\vec{\theta}}{4} + \frac{P^2\theta^2 - (\vec{P}.\vec{\theta})^2}{16}}{l^2} + \frac{J^2}{4\left(r_{h\pm}^2 - \frac{\vec{L}.\vec{\theta}}{4} + \frac{P^2\theta^2 - (\vec{P}.\vec{\theta})^2}{16}\right)} = 0 \tag{18}$$

where $L_k = \varepsilon_{ijk}x_i p_j$, $p^2 = \vec{p}.\vec{p}$ and $\theta^2 = \vec{\theta}.\vec{\theta}$. We can rewrite Eq. (18) as

$$r^4 - \left(Ml^2 - \frac{\vec{L}.\vec{\theta}}{2} - \frac{P^2\theta^2 - (\vec{P}.\vec{\theta})^2}{8}\right) r^2 - \left(Ml^2\vec{L}.\vec{\theta} + Ml^2\frac{(P^2\theta^2 + (\vec{P}.\vec{\theta})^2)}{16} - \frac{(\vec{L}.\vec{\theta})^2}{16}\right) = 0 \tag{19}$$

therefore the horizons of BTZ black hole in non-commutative space are as following

$$r_{h\pm} = l\left[\frac{M}{2}\left(1 - \frac{\vec{L}.\vec{\theta}}{Ml^2} - \frac{P^2\theta^2 - (\vec{P}.\vec{\theta})^2}{8Ml^2} \pm \sqrt{1 - \left(\frac{J}{Ml}\right)^2}\right)\right]^{1/2} \tag{20}$$

For rotating BTZ black hole, in the limit $\theta \to 0$, the horizons are reduced to one in commutative space. For the non-rotating BTZ black hole, the horizons are as following

$$r_{h+} = l\left[M - \frac{p^2\theta^2 - (\vec{P}.\vec{\theta})^2}{16l^2}\right]^{1/2} \tag{21}$$

$$r_{h-} = \frac{[p^2\theta^2 - (\vec{P}.\vec{\theta})^2]^{1/2}}{4} \tag{22}$$

It is interesting that in non-commutative case, non-rotating BTZ black hole in contrast with commutative case has two horizons. As one can see in non-rotating case, to the first order of perturbative calculations, there is no any effect on horizon r_+ due to the non-commutativity of space, and $r_- = 0$.

The horizon area and Hawking temperature of BTZ black hole in non-commutative space are respectively as

$$\begin{aligned} A &= 2\pi r_{h+} \\ &= 2\pi l \left[\frac{M}{2} \left(1 - \frac{\vec{L}.\vec{\theta}}{Ml^2} - \frac{P^2\theta^2 - (\vec{P}.\vec{\theta})^2}{8Ml^2} + \sqrt{1 - \left(\frac{J}{Ml}\right)^2} \right) \right]^{1/2} \end{aligned} \tag{23}$$

$$\begin{aligned} T_H &= \frac{1}{2\pi l^2} \left(\frac{r_{h+}^2 - r_{h-}^2}{r_{h+}} \right) \\ &= \frac{\sqrt{1 - \left(\frac{J}{Ml}\right)^2}}{\pi \left[\frac{M}{2} \left(1 - \frac{\vec{L}.\vec{\theta}}{Ml^2} - \frac{P^2\theta^2 - (\vec{P}.\vec{\theta})^2}{8Ml^2} + \sqrt{1 - \left(\frac{J}{Ml}\right)^2} \right) \right]^{1/2}} \end{aligned} \tag{24}$$

4. CONCLUSION

In this paper we have examined the effects of the space non-commutativity on the properties of rotating and non-rotating BTZ black holes. The event horizon of the black hole undergoes corrections from the non-commutativity of space as Eq. (20) for rotating case, and Eqs. (21, 22) for non-rotating case. It is interesting that in non-commutative case, non-rotating BTZ black hole in contrast with commutative case has two horizons. Since the non-commutativity parameter is so small in comparsion with the length scales of the system, one can consider the noncommutative effect as perturbations of the commutative counterpart (Li, in press). Then we have obtained the corrections to the temperature and entropy as Eqs. (23, 24).

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